Minimal surfaces, spatial topology and singularities in space-time

1983 J. Phys. A: Math. Gen. 16 1435

(http://iopscience.iop.org/0305-4470/16/7/019)

J. Phys. A: Math. Gen. 16 (1983) 1435–1439. Printed in Great Britain

Minimal surfaces, spatial topology and singularities in space–time

Gregory J Galloway
Department of Mathematics, University of Miami, Coral Gables, FL 33124, USA

Received 3 September 1982

Abstract. Using an approach first advocated by Gannon and recent results of Meeks, Simon and Yau on the existence of compact minimal surfaces, some new results are obtained relating non-trivial spatial topology to the occurrence of singularities in space–time. For example, it is shown that if V^3 is a contracting body with mean-convex boundary homeomorphic to a two-sphere in a space–time M^4 obeying appropriate curvature and causality assumptions, then either V^3 is a three-cell or M^4 is non-space-like geodesically incomplete.

1. Introduction

Gannon (1975, 1976), Lee (1976) and Lindblom and Brill (1980) have obtained results which, in one view, establish restrictions on the spatial topology of non-singular (i.e. geodesically complete) space–times. These results by and large require a number of stringent global assumptions (the assumption of completeness being one such condition). More recently, Frankel and Galloway (1982) have obtained restrictions on the spatial topology from purely *local* assumptions. However, because of the local nature of the hypotheses, these results do not allow an interpretation which is allowed by the aforementioned results, namely, that non-trivial spatial topology leads to the occurrence of singularities. As an example consider the following result of Gannon (1975).

Theorem (*Gannon 1975*). Let M^4 be a space–time which satisfies the null convergence condition and admits a Cauchy surface V^3 which is regular near infinity. If V^3 is non-simply connected then M^4 is null geodesically incomplete.

(The assumption that V^3 be regular near infinity requires that it be asymptotically flat in some appropriate sense.)

The purpose of this note is to present some new results of this type which improve various aspects of previous results. Most importantly, we obtain results which do not require that M^4 be globally hyperbolic or, in particular, that V^3 be Cauchy. These conditions are also avoided in the statements of theorems 2.1 and 2.2 in Gannon (1975). Unfortunately, Gannon's proofs of these theorems rely crucially on an observation (p 2366, paragraph 2 of his paper) which we have recently discovered to be false. The results presented here recover in spirit some aspects of Gannon's theorem 2.1, and improve other aspects. Our main results apply to a space-like hypersurface

V^3 (perhaps with boundary) which need not be edgeless or acausal. Furthermore, the assumption of non-simple connectivity is substantially weakened. Basically, it is only required that the topology of V^3 be non-Euclidean. On the other hand, it will be necessary to impose certain *extrinsic* conditions on V^3.

The approach taken here is that advocated by Gannon (1975, p 2367). Under the assumption that a suitable space-like hypersurface V^3 has non-trivial topology, we make use of the recent results of Meeks *et al* (1982) concerning the existence of compact minimal surfaces to obtain a compact minimal surface W^2 in V^3. Under suitable extrinsic conditions on V^3 it is shown that W^2 is a closed trapped surface and, hence, by standard singularity theorems, one is able to conclude that space–time is non-space-like geodesically incomplete.

2. The results

By a *body* V^3 we mean a compact three-dimensional Riemannian manifold with smooth boundary ∂V (which need not be connected). We will say that ∂V is *mean-convex* if its mean curvature with respect to the outer normal is non-negative, $H(\partial V) \geqslant 0$. (We are using the sign convention currently popular in relativity in which $H \geqslant 0$ implies that the outer normal is *diverging* on the average at each point of ∂V. Gannon (1975) and Lee (1976) have considered a related but different convexity condition.)

Let M^4 denote an arbitrary space–time, by which we mean a smooth four-dimensional time oriented Lorentzian manifold having signature $(-+++)$. We will consider only those bodies V^3 embedded as space-like submanifolds of M^4 carrying the induced metric.

To establish the existence of a closed trapped surface it will be necessary to impose certain *extrinsic* conditions on V^3. Let Z be the future pointing unit normal vector field along a space-like hypersurface V^3 (perhaps with boundary). Introduce the second fundamental form (or expansion tensor) Θ of V^3 as follows,

$$\Theta(X, Y) = \langle \nabla_X Z, Y \rangle$$

for vectors X, Y tangent to V, where $\langle , \rangle$ is the space–time metric and ∇ is the associated Levi–Civita connection. (See e.g. Frankel (1979) for a discussion of the kinematical and geometrical significance of Θ.) At any point of V^3 let θ be the trace of the linear map $X \to \nabla_X Z$ associated with the bilinear form Θ; for any extension of Z to a time-like unit vector field in a space–time neighbourhood of V, $\theta = \operatorname{div} Z$. If $\theta < 0$ along V^3 then, at each point p of V, V is contracting *on the average* over all directions in V at p. This condition occurs in several of the singularity theorems of general relativity (see e.g. Hawking and Ellis 1973, theorem 4, p 272). In these results V^3 is globally defined (e.g. is Cauchy or compact *without* boundary). In the main results to be presented here it will be necessary to impose a more stringent contraction assumption which, however, need only hold locally, i.e. hold on a body V^3. It will be required that the body V^3 be contracting in *all* directions.

A space-like hypersurface V^3 is said to be contracting in all directions or non-expanding in all directions if Θ is negative definite or negative semi-definite, respectively, at each point of V^3. (Note: if X is a unit tangent vector to V^3 which is extended by making it invariant under the normal geodesic flow then $\Theta(X, X) = \mathrm{d}/\mathrm{d}s\|X\|$, where s is proper time and $\|X\| = \langle X, X \rangle^{1/2}$.)

The following lemma gives an interesting geometric realisation of closed trapped surfaces in space–time. Recall that a smooth surface W^2 in V^3 is said to be minimal if its mean curvature vanishes.

Lemma. Let $V^3 \subset M^4$ be a space-like hypersurface (possibly with boundary) which is contracting in all directions. Let W^2 be a compact minimal surface in V^3. Then W^2 is a closed trapped surface in M^4.

Proof. Each sufficiently small piece of W^2 admits a smooth non-vanishing unit normal N tangent to V^3 (although there may not exist such a non-vanishing normal defined globally along W^2). Let Z be the unit time-like future pointing normal to V^3. Then the equation

$$K_{\pm} = Z \pm N$$

defines locally two non-vanishing null vector fields orthogonal to W^2. Introduce the two null expansion tensors (or null second fundamental forms) $\chi_{\pm}$ by

$$\chi_{\pm}(X, Y) = \langle \nabla_X K_{\pm}, Y \rangle,$$

where X and Y are tangent to W^2. To show that W^2 is trapped it suffices to show that $\operatorname{Tr} \chi_{\pm} < 0$, i.e. that the two families of null geodesics issuing orthogonally from W^2 in the directions K_+ and K_- are converging. (By $\operatorname{Tr} \chi_{\pm}$ we mean the trace of the associated linear transformation $X \rightarrow \nabla_X K_{\pm}$.)

With respect to an orthonormal basis $\{e_1, e_2\}$ of the tangent space of W^2 at a given point,

$$\operatorname{Tr} \chi_{\pm} = \sum_{\alpha=1}^{2} \langle \nabla_{e_\alpha} K_{\pm}, e_\alpha \rangle = \sum_{\alpha=1}^{2} \langle \nabla_{e_\alpha} Z, e_\alpha \rangle \pm \sum_{\alpha=1}^{2} \langle \nabla_{e_\alpha} N, e_\alpha \rangle$$

$$= \sum_{\alpha=1}^{2} \Theta(e_\alpha, e_\alpha) \pm H < 0,$$

since Θ is negative definite and H = mean curvature of $W^2 = 0$.

Meeks *et al* (1982) have recently obtained some definitive results concerning the existence of compact minimal surfaces in Riemannian three-manifolds. We draw from their results only what is needed in the present study. Recall that a handlebody is a diffeomorphic copy of a body in $\mathbb{R}^3$ which is bounded by a smooth surface of genus g. It is worth keeping in mind that the boundary of a handlebody must be connected, and a handlebody whose boundary is topologically a two-sphere must be a three-cell.

Theorem (Meeks, Simon, Yau). Let V^3 be a body with mean-convex boundary. If V^3 is not a handlebody then V^3 contains a compact minimal surface W^2.

The two previous results, together with the Penrose singularity theorem (see Hawking and Ellis 1973, theorem 1, p 263), yield the following singularity result.

Theorem 1. Let M^4 be a space–time which admits a non-compact Cauchy surface and satisfies the null convergence condition, $\operatorname{Ric}(K, K) = R_{ij}K^iK^j \geqslant 0$ for all null vectors K. Suppose that V^3 is a body contracting in all directions with mean-convex boundary. If V^3 is not a handlebody then M^4 is future null geodesically incomplete.

Proof. By the results of Meeks, Simon and Yau, V^3 contains a compact minimal surface W^2. By the lemma, W^2 is a closed trapped surface. By the Penrose singularity theorem, M^4 is future null geodesically incomplete.

Remarks

(1) The hypotheses of theorem 1 are satisfied in the black hole region of the Schwarzschild solution. With respect to the standard Schwarzschild coordinates r, t, θ, ϕ, one can find bodies (or, in this case, 'wormholes') of the form $r=f(t)$, $-T \leq t \leq T$, with all the requisite properties. (Recall that t is a space-like coordinate and r is a time-like coordinate in the black hole region.) In particular, since the boundary of such a body consists of two disjoint two-spheres, it is not a handlebody. We note, however, that the surfaces $r=\text{constant}$ do not meet all the requirements. Although these surfaces are contracting in directions tangent to the two-spheres $t=\text{constant}$, they are expanding along the longitudes $\theta=\text{constant}$, $\phi=\text{constant}$. In fact, for $m<r<2m$, the *mean* expansion of each of these surfaces is positive.

(2) The assumption that V^3 be contracting in all directions can be replaced by the slightly weaker assumption that it be non-expanding in all directions provided it is assumed that the curvature tensor obeys the generality condition: $K^cK^dK_{[a}R_{b]cd[e}K_{f]} \neq 0$ at some point along each future directed null geodesic η issuing from V^3, where $K=K^a\partial_a$ is the tangent to η. Indeed, in this case, one can still establish the occurrence of a focal point to W^2 along each future directed null geodesic issuing orthogonally from W^2 (assuming, by contradiction, that M^4 is future null geodesically complete), as is needed in the proof of the Penrose singularity theorem.

(3) Let $M=\text{Minkowski space}$ and let V be the body described by $t=-(1+x^2+y^2+z^2)^{1/2}$, $x^2+y^2+z^2 \leq 1$. This example shows that theorem 1 is false if the assumption that V^3 not be a handlebody is dropped.

In fact, by restricting attention to bodies having boundaries which are topologically two-spheres, theorem 1 immediately yields the following.

Corollary 2. Assume M^4 satisfies the null convergence condition and admits a non-compact Cauchy surface. Suppose that V^3 is a body contracting in all directions having a mean-convex boundary which is homeomorphic to a two-sphere. Then, either V^3 is a three-cell or M^4 is future null geodesically incomplete.

The following result applies to space-like hypersurfaces without boundary.

Corollary 3. Assume M^4 satisfies the null convergence condition and admits a non-compact Cauchy surface. Let $V^3 \subset M^4$ be a space-like hypersurface contracting in all directions such that $V^3=\bigcup_{i=1}^{\infty} V_i$, where, for each i,

(1) V_i is a body having mean-convex boundary ∂V_i which is topologically a two-sphere, and

(2) $V_i \subset V_{i+1}-\partial V_{i+1}$.

Then either V^3 is homeomorphic to $\mathbb{R}^3$, or M^4 is future null geodesically incomplete.

We remark that V^3 need not be edgeless or acausal.

Proof. If some V_i is not a three-cell then, by theorem 1, M^4 is future null geodesically incomplete. Suppose, then, that each V_i is a three-cell. Then $V_i-\partial V_i$ is an open three-cell and $V^3=\bigcup_{i=1}^{\infty}(V_i-\partial V_i)$. But a topological space which is an increasing union of open three-cells must itself be an open three-cell (see Brown 1961). Thus, V^3 is homeomorphic to $\mathbb{R}^3$.

By calling upon the Hawking–Penrose singularity theorem (see Hawking and Ellis 1973, theorem 2, p 266) and strengthening the curvature assumption somewhat, we can replace the assumption of the existence of a non-compact Cauchy surface by the chronology condition.

Theorem 4. Consider a space–time M^4 which satisfies the null and time-like convergence conditions, the generic condition and the chronology condition. Suppose that V^3 is a body contracting in all directions with mean-convex boundary. If V^3 is not a handlebody then M^4 is non-space-like geodesically incomplete.

Proof. Similar to theorem 1.

Although the statements are omitted here, theorem 4 implies the obvious analogues of corollaries 2 and 3. Furthermore, remark 2 following theorem 1 applies to theorem 4, as well.

It would perhaps be worthwhile investigating to what extent the results presented here rely on the contraction assumption. The results of Lee (1976) and Gannon (1975), although they only apply to globally hyperbolic space–times, suggest that such extrinsic conditions may be substantially weakened.

References

Brown M 1961 *Proc. Amer. Math. Soc.* **12** 812–4
Frankel T 1979 *Gravitational curvature* (San Francisco: Freeman)
Frankel T and Galloway G 1982 *Math. Zeit.* **181** 395–406
Gannon D 1975 *J. Math. Phys.* **16** 2364–7
—— 1976 *Gen. Rel. Grav.* **7** 219–32
Hawking S W and Ellis G F R 1973 *The large scale structure of the universe* (Cambridge: CUP)
Lee C W 1976 *Commun. Math. Phys.* **51** 157–62
Lindblom L and Brill D R 1980 *Comments on the topology of nonsingular stellar models* in *Essays in general relativity* ed F J Tipler (New York: Academic)
Meeks W, Simon L and Yau S-T 1982 *Ann. Math.* **116** 621–59